CHAPTER 6
Dominance & Best Response
Goals

- Notion of “dominated strategy” as one that is worse than another, whatever others turn out to do
- A notion of rationality: “do not play dominated strategies”
- In Prisoners’ Dilemma, keeping quiet is dominated
- Tension: Individual optimality ≠ Group optimality
- A notion of group optimality
- Limitation of rationality notion based on “dominated strategies”
- Alternative notion: best response strategy to a given belief
- Relation between two notions of rationality
Dominated strategies: D an example

- Note 1’s strategy D: Regardless of what 2 does, 1’s payoff under D < 1’s payoff under U when 1<2 (if 2 plays L) and 4<5 (if 2 plays R).

Note 1’s strategy D: Regardless of what 2 does, 1’s payoff under D < 1’s payoff under M when 3<4 (if L), 0<1 (if C) & 2<5 (if R).

Focus on D again. No longer any strategy with 1’s payoff under D < 1’s payoff under strategy. But: --‘’< 1’s payoff under σ=(1/2,1/2,0), for 1<(1/2)4+(1/2)0 (if L) & 1<(1/2)0+(1/2)4.
Dominated strategies: D an example

• Let $s_i$ be a strategy of player $i$. Say it is dominated if there is another strategy (pure or mixed) $\sigma_i \in \Delta S_i$ such that

$$u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})$$

for every strategy profile $s_{-i} \in S_{-i}$ of the other players. A dominated strategy is worse not just in expectation, but whatever others do!

• Referring to previous slide, in every game D is dominated:
  - Game 1: D is dominated by U, which is $\sigma=(1,0)$
  - Game 2: D is dominated by M, which is $\sigma=(0,1,0)$
  - Game 3: D is dominated by $\sigma=(1/2,1/2,0)$

  Btw: In Game 2, for player 2, R is not dominated by C
Dominated strategy? Yes

There is another strategy $\sigma = s'$ to cut a branch that avoids certain injury, as in his strategy $s$. 

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Dominated strategy? No

- Julio Aparicio had other strategies that day, e.g. stay home
- Maybe these entailed sure, large costs: loss of position, fans...
- While with his strategy, getting gored was not guaranteed
- So his strategy was not dominated (had the bull strategized differently, Julio would have remained fine and beloved)
Dominated strategies & rationality

• There are many senses of rationality, a minimalistic one is: “Never play a dominated strategy”

• Why? Because there is some other strategy that secures a higher payoff (not just in expectation)

  Note: “for every” in the definition of “dominated” is key

• Let us apply this to the Prisoner’s Dilemma:
  For player 1: C dominated by D
  For player 2: C dominated by D

  ... so, if players 1,2 are minimally rational, neither plays C, hence both play D & get payoff (1,1)

• **Shocking**: Though both rational, (2,2) payoff would be better
Individual rationality X
Group rationality

• The Prisoners’ Dilemma illustrates a sad fact of life

• Namely, actions that may be rational/best from an individual perspective can be irrational/poor from a group’s perspective

• Individual rationality leads to poor payoffs (1,1)
Group rationality (whatever this means) would never lead to payoffs (1,1), since (2,2) are superior and available

• Next, we introduce a notion of group rationality. It does allow for multiple strategy profiles to qualify as rational.
Efficiency – “social rationality”

• Let $s'$, $s$ be strategy profiles (listing strats for all players)
• Say $s'$ is more efficient than $s$ (or Pareto superior to) if
  \[ u_i(s') \geq u_i(s) \]
  for every player $i$, with strict $>$ inequality for at least one.
• Example: (C,C) is more efficient than (D,D) in the PD.

• Say a strategy profile $s$ is efficient if there exists no other strategy profile that is more efficient than $s$.
• Example: (C,C), (C,D), (D,C) are all efficient in the PD. (D,D) is not owing to (C,C) = $s'$.
• Summarizing Prisoners’ Dilemma: (D,D) rational, not efficient.
Dominated strategies: often no bite

- Recall, notion of dominated strategies useful b/c allows us to define minimal rationality as “do not play dominated strats”
- Seems compelling, but has no bite when player has no dominated strats. For then definition becomes “do not play ...(silence)…”

- Example: In this game,
  - for player 1: H is not dominated
  - for player 2: H is not dominated
  - T is not dominated
- Need notion of rationality with more bite, for such games
Intermission

- Introduced notion of “dominated strategy” then suggested a notion of rationality as “do not play any dominated strategy”
- Notion is compelling in those games where it implies a unique prediction, as in the Prisoner’s Dilemma.
- But in some games it implies nothing, as in Matching Pennies
- For those games, want another notion of rationality
- Will now introduce one that is relative to beliefs. Thus given a player’s belief, we will define whether a strategy is rational. A strategy can be rational for one belief but not for another!
- In contrast, “do not play any dominated strategy” is rational for any belief of the player.
Review of expected payoff

- Fix a normal form game and a player, \( i \)
- Given her belief \( \theta_{-i} \) about the other players’ strategies, her expected payoff of playing strategy strategy \( s_i \)

\[
u_i(s_i, \theta_{-i}) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) u_i(s_i, s_{-i})
\]

Example: \( i=1, \theta_{-1}=(1/3,1/2,1/6) \)

\[
u_1(U, \theta_{-1})= (1/3)2+(1/2)0+(1/6)4=8/6
\]
\[
u_1(M, \theta_{-1})= (1/3)3+(1/2)0+(1/6)1=7/6
\]
\[
u_1(D, \theta_{-1})= (1/3)1+(1/2)3+(1/6)2=13/6
\]

Given this belief, \( 1 \)’s “best” strategy is \( D \).
Best response to a belief

• Natural notion of rationality given belief about others’ strats: i plays a strategy that maximizes her payoff \( u_i \) (see slide 1):

• Suppose player \( i \) holds belief \( \theta_{-i} \in \Delta S_{-i} \) about others’ strategies. Then i’s strategy \( s_i \in \Delta S_i \) is called a best response to belief \( \theta_{-i} \) if

\[
 u_i(s_i, \theta_{-i}) \geq u_i(s_i', \theta_{-i})
\]

for every other possible strategy \( s'_i \) of player \( i \).

Example: In slide 1, for player 1, D is a best response to belief \( \theta_{-1} = (1/3, 1/2, 1/6) \). Neither U nor M is.
Best response to a belief – con’t

• Write $\text{BR}_i(\theta_{-i})$ for the set of i’s best responses to belief $\theta_{-i}$.

• Previously, $\text{BR}_1((1/3,1/2,1/6)) = \{D\}$, a singleton

• Multiple best responses are possible. Turn to player 2, belief $\theta_{-2} = (1/2,1/4,1/4)$

$u_2(L, \theta_{-2}) = (1/2)6+(1/4)3+(1/4)1=4$
$u_2(C, \theta_{-2}) = (1/2)4+(1/4)0+(1/4)5=13/4$
$u_2(R, \theta_{-2}) = (1/2)4+(1/4)5+(1/4)3=4$

Here, $\text{BR}_2((1/2,1/4,1/4)) = \{L,R\}$, multiple
Rationality speaks of both actions & beliefs

Proposed notion of rationality relative to a belief.

What if the belief is “irrational,” i.e. wrong statistically, so other players play differently than believed?

Then the expected payoff (under the statistical probability) of one’s “rational” strategy (best response to one’s wrong belief) may be quite low.

Rationality entails “rational belief,” not just “rational strategy”

More on “rational beliefs” when we introduce “Nash equilibrium” ... for now we focus on “rational strategies” (i.e. on best responses)
Best response strategy X
Undominated strategy

Got two notions of rationality for a player:
- play only un-dominated strategies
- given a belief, play a best response to it

How are they related?

**Theorem**  For any finite game and strategy for i, it is a best response to some belief IFF it is undominated

(Or: it is dominated $\iff$ for no belief is it a best response)

• Argument for why, if dominated, not a b.r. to any belief: Say $s_i$ is dominated by $\sigma_i$ & let $\theta$ be any belief. Multiplying slide 4’s inequality by $\theta(s_{-i})$ and adding up over $s_{-i}$ preserves inequality, hence $u_i(s_i, \theta_{-i}) < u_i(\sigma_i, \theta_{-i})$ which says $s_i$ is worse than $\sigma_i$ hence not a b.r. to this (arbitrary) belief
Restatement in terms of sets

- Let $B_i = \{ s_i \mid s_i \in BR_i(\theta_{-i}) \text{ for some belief } \theta_{-i} \in \Delta S_{-i} \}$
  Those strategies that are best responses to some belief

- Let $UD_i = \{ s_i \mid s_i \text{ is not dominated} \}$

- **Theorem** For any finite game and player $i$, $B_i = UD_i$
Interpretation

• If a strategy is undominated, it could be rationalized as a best response, but possibly relative to a very crazy, odd belief.

• Conversely, however crazy and odd a player’s belief is, so long as the player is best responding to this belief, this best response strategy will show the craziness does have a limit: *any other strategy*, in some special circumstance (i.e. for some profile of others’ strategies), will pay off the same as or worse than this best response.
Beliefs in correlated strategies

• For its validity, the theorem relies on beliefs of the sort $\theta_{\cdot i} \in \Delta S_{\cdot i}$, so called “correlated beliefs”

• Such beliefs entertain the possibility that other players coordinate/correlate their strategies. Such beliefs are not restricted to assigne, player by player, a probability of what the player’s strategy is

• E.g.: With 3 players, 1 may believe that 2,3 will either both attack or run away, and never “one attacks, other runs away”. Such a belief cannot be split into one about 2, one about 3.

• In the case of two players, this distinction is moot, since “other players” is just one, and coordination presumes more than one
Beliefs in uncorrelated strategies

• Here are beliefs that entertain only uncorrelated strategies:

\[ B^u_i = \{ s_i | s_i \in BR_i(\theta_{-i}) \text{ for some belief } \theta_{-i} = p_1 x \ldots p_{i-1} x p_{i+1} x \ldots p_n \} \]

These are products of beliefs about individual opponents.

• If we so restrict beliefs, part of theorem survives:

• Proposition  \( B^u_i \) is contained in \( UD_i \)

  Given a strategy for \( i \), if it is a best response to some uncorrelated belief, then it is undominated.