



# A social welfare function characterizing competitive equilibria of incomplete financial markets

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## ABSTRACT

A classic characterization of competitive equilibria views them as feasible allocations maximizing a weighted sum of utilities. It has been applied to establish fundamental properties of the equilibrium notion, such as existence, determinacy, and computability. However, it fails for economies with missing financial markets.

We give such a characterization for economies with a single commodity and missing financial markets, by an amended social welfare function. Its parameters capture both the relative importance of households welfare, through the classic welfare weights, as well as the disagreements among them as to the value of the missing markets.

As a by-product, we identify the dimension of the set of interior equilibrium allocations.

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## 1. Introduction

If no financial markets are missing, following Lange (1942) and Allais (1943), interior allocations of given resources  $r$  are competitive equilibria if and only if they solve the program  $\max_{\Sigma x^h=r} W_\delta(x)$  for some strictly positive  $\delta$ , with  $W_\delta$  being the social welfare function<sup>1</sup>

$$W_\delta(x) := \Sigma \delta^h u^h(x^h). \quad (1)$$

The parameters  $\delta$  in Lange's social welfare function capture the relative importance of households' welfare. This characterization has been applied to establish fundamental properties of the equilibrium notion: existence, Negishi (1960) and Bewley (1969), determinacy with infinitely long living households, Kehoe and Levine (1985), and computability, Mantel (1971).

If some financial markets are missing as in Radner (1972), however, this equivalence fails: some interior competitive equilibria need not solve the program  $\max_{\Sigma x^h=r} W_\delta(x)$  for any strictly positive  $\delta$ . Moreover, no natural social welfare function  $W$  has been found that would rescue this implication.

We extend the characterization to economies with some missing financial markets, by amending the social welfare function. Thus interior allocations of given resources are competitive equilibria if and only if they solve the program  $\max_{\Sigma x^h=r} W_{\delta,\mu}(x)$  for some parameters  $\delta \in \mathbb{D}$ ,  $\mu \in \mathbb{M}$  living in certain spaces, with  $W_{\delta,\mu}$  being the social welfare function

$$W(x) := \Sigma \delta^h u^h(x^h) - \Sigma \mu^h \cdot x_1^h. \quad (2)$$

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<sup>1</sup> Lange characterizes Pareto optima in this way. So the above characterization follows from the two welfare theorems. (Lange (1942) is aware of the first one, while Allais (1943) is among the first to rigorously prove the second one.)

Here, the *social evaluation* of allocations is described by the usual weights  $\delta$  on households' welfare, and by new charges  $\mu$  on their future consumption. The parameter  $\delta$  is interpreted classically, whereas  $\mu$  is interpreted as the “disagreement” among households as to the “value” of the “missing financial markets”, as justified below.

Why does it fail, the equivalence of competitive equilibria and maxima of (1), if some financial markets are missing? On the one hand, any allocation  $x$  that maximizes this is Pareto efficient. Indeed, if  $y$  were Pareto superior to  $x$ , i.e.  $(u^h(y^h)) > (u^h(x^h))$ , then  $W_\delta(y) > W_\delta(x)$  for any  $\delta \gg 0$ , so  $x$  could not be a maximum for any  $\delta \gg 0$ . On the other hand, some allocations  $x$  that are competitive equilibria of incomplete financial markets are Pareto inefficient. Indeed, for almost every initial allocation, every competitive equilibrium allocation is Pareto inefficient; for an exposition of this well-known fact, see Magill and Quinzii (1996).<sup>2</sup> So some competitive equilibria fail to maximize (1) for any  $\delta \gg 0$ .

We explain in what sense the parameter  $\mu$  is the “disagreement” among households as to the “value” of the “missing financial markets”, by clarifying each of these terms. By “missing financial markets” we mean the orthogonal complement  $a^\perp$  of the span of the existing financial instruments  $a$ . By “value” of the missing financial markets we mean a linear functional  $v : a^\perp \rightarrow \mathbb{R}$ . The separating hyperplane theorem implies that any linear functional on a finite-dimensional vector space can be represented uniquely as the inner product against a unique element of the vector space—call this element  $\hat{v} \in a^\perp$ , so that  $v(m) = m \cdot \hat{v}$ . If each household  $h$  thinks such a *value*  $v^h$ , the *disagreement* is then the differences from the mean,  $\mu^h := \hat{v}^h - \text{mean}(\hat{v}^1, \dots, \hat{v}^H)$ . When so defined, the disagreement  $\mu = (\mu^h)$  satisfies two properties: (i)  $\mu^h \in a^\perp$ , because it is a linear combination of points  $\hat{v}^h \in a^\perp$  in a vector space, and (ii)  $\Sigma \mu^h = 0$ , because these are differences from the mean. In sum, imagining that each household has its own  $v^h$ , an opinion as to the value of the missing financial markets, then this is the sense of the new parameter in our social welfare function (1)—a matrix  $\mu = (\mu^h)$  satisfying conditions (i), (ii).

Our main contribution is a fine characterization of the set  $\mathbb{X}$  of interior competitive equilibrium allocations of all the economies parameterized by initial endowment distributions  $(e^h)_h$  of given state-contingent, aggregate resources  $r$ . In doing so, besides  $r$ , we take as given smooth preferences  $u$  as in Debreu (1972, 1976), and the asset structure formed by finitely many financial instruments  $a$ . The characterization is accomplished in steps, establishing three results.

The **first result** (Theorem 1) is that an allocation  $x \gg 0$  is an equilibrium allocation if and only if it solves the program  $\max_{\Sigma x^h=r} W_{\delta, \mu}(x)$  for some  $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$ , where

$$\mathbb{D} := \left\{ \delta \in \mathbb{R}^H \mid \delta \gg 0, \Sigma \frac{1}{\delta^h} = 1 \right\} \quad \mathbb{M} := \left\{ \mu \in (a^\perp)^H \mid \Sigma \mu^h = 0 \right\}.$$

We see that the “welfare” parameter  $\delta$  is normalized in a standard way, and the “disagreement” parameter  $\mu$  reflects properties (i) and (ii) above.

The **second result** (Proposition 1, part A) identifies the  $(\delta, \mu)$  from the equilibrium allocation as being

$$\delta^h(x) = \frac{1}{D_{x_0} u^h(x^h)} \tag{3}$$

$$\mu^h(x) = \hat{v}^h - \text{mean}(\hat{v}^1, \dots, \hat{v}^H) \quad \text{with } \hat{v}^h := \frac{D_{x_1} u^h(x^h)}{D_{x_0} u^h(x^h)}.$$

Thus  $\delta^h$  is the inverse of the marginal utility of present consumption, as usual, and  $\mu$  is, as interpreted above, the disagreement among households as to the value of the missing financial markets, where each household's “value”  $\hat{v}^h$  is concretized as the marginal rates at which it substitutes consumption in future states for consumption in the present state. Here, the abstract notion of “value” as a linear functional  $v : a^\perp \rightarrow \mathbb{R}$  is made concrete by the idea of marginal willingness to pay as  $\Delta \mapsto \Delta \cdot \text{MRS}$ , the inner product of the infinitesimal change  $\Delta$  in future consumption against the marginal rates of substitution *MRS*.

The **third result** refines the first two. Theorem 2 establishes that the relation  $x \leftrightarrow (\delta, \mu)$  between  $\mathbb{X}$  and  $\mathbb{D} \times \mathbb{M}$  is a bijection, smooth in both directions. This implies immediately that the dimension of  $\mathbb{X}$  equals the dimension of  $\mathbb{D} \times \mathbb{M}$ , which is easily shown to be  $(H - 1)(1 + m)$ , where  $m$  is the number of missing financial markets. This nests a well-known fact about complete markets, where  $m = 0$ : the interior Pareto optima (which are  $\mathbb{X}$  by the two welfare theorems) have dimension  $H - 1$ ; see proof 5.2.4 in Balasko (1988).

We focus attention on an exchange economy that has a single good per state and in which asset payoffs are denominated in the numéraire. Although restrictive, this context is interesting both theoretically and for financial applications. Theoretically, because it allows one to concentrate on financial markets, leading aside issues concerning spot markets. As for applications, a single good suffices to embody, in a general equilibrium model, classical models of asset pricing such as the CAPM.

Extensions of our characterization to economies with multiple goods and spot markets are substantially available in two other works, Siconolfi and Villanacci (1991) and Tirelli (2008), whose understanding of the geometry of the equilibrium set is instrumental, respectively, in the study of the indeterminacy and of the welfare properties of equilibria also in the sense of

<sup>2</sup> If there are multiple goods and enough missing financial markets, even the equilibrium use of the existing financial markets is generically Pareto inefficient, as shown by Geanakoplos and Polemarchakis (1986), who pioneer the application of transversality to equilibrium welfare. The intuition for this is due to Stiglitz (1982).

Geanakoplos and Polemarchakis (1986). With respect to these papers, our single-good characterization is simpler, proposing a welfare function that differs from the complete market analogue,  $W_\delta(x)$ , only by a term that is linear in the allocation  $x$ . As often happens, a greater level of simplicity implies a loss of generality. Yet, the methodology carefully spelled out in this paper to derive our equilibrium characterization extends to other possible formalizations of the welfare function, such as those proposed in the existing literature.

The rest of the paper is organized as follows. Section 2 spells out the model and assumptions. Section 3 develops the characterization. Section 4 refines the characterization, computing the dimension of  $\mathbb{X}$ . Section 5 contains the more formalistic and less insightful proofs.

## 2. Economy and equilibria

*Primitives.* There are  $h = 1, \dots, H$  households who know the present state of nature 0 but are uncertain as to which future state of nature  $s = 1, \dots, S$  will occur. In each state a nonstorable commodity is available for consumption, and in state 0 there are  $j = 1, \dots, J$  assets available for trade.

*Economy.* Given fixed aggregate resources  $r \in \mathbb{R}_{++}^{S+1}$  of the commodities,  $e \in \mathbb{R}_{++}^{H(S+1)}$  specifies a distribution of  $r$  across households, and  $\Omega := \{e \in \mathbb{R}_{++}^{H(S+1)} : \Sigma e^h = r\}$  the set of all feasible distributions.<sup>3</sup> Each asset  $j$  pays off a state-contingent amount  $a^j \in \mathbb{R}^S$  of the commodity in the future;  $a \in \mathbb{R}_{++}^{S \times J}$  denotes the asset payoff matrix. Asset markets are **complete** if  $\text{span}(a) = \mathbb{R}^S$ , and **incomplete** otherwise.

*Markets.* Markets specify that each asset  $j$  is tradeable at a price of  $q^j$  units of the commodity in the present, by specifying  $q = p'a$  (row) for some state prices  $p \in \mathbb{R}_{++}^S$ .  $Q \subset \mathbb{R}^J$  denotes such asset prices. Households are free to trade any amount  $\theta_j^h \in \mathbb{R}$  of any asset: buy,  $\theta_j^h > 0$ ; sell,  $\theta_j^h < 0$ ; or neither,  $\theta_j^h = 0$ . Trades of asset  $j$  **clear** if  $\Sigma \theta_j^h = 0$ . Viewing asset prices as a negative present claim, asset claims become  $W := \begin{pmatrix} -q \\ a \end{pmatrix} \in \mathbb{R}^{(S+1) \times J}$ .

**Remark 1.** The asset claims  $a$  and resources  $r$  are fixed for the rest of the paper. This is important in interpreting the dimensions reported in Section 4.

*Consumption and asset trade.* The **consumption correspondence** is a mapping of asset prices and own endowment,  $X : Q \times \mathbb{R}_{++}^{S+1} \rightarrow \mathbb{R}_{++}^{S+1}$ , such that

$$X(q, e^h) = \text{span}(e^h + W\mathbb{R}^J) \cap \mathbb{R}_{++}^{S+1}.$$

Asset **trade** is a function of asset prices and own endowment,  $\theta : Q \times \mathbb{R}_{++}^{S+1} \rightarrow \mathbb{R}^J$ . Each household with an asset trade  $\theta^h$  consumes  $x^h(q, e^h) := e^h + W\theta^h(q, e^h)$ . Trades are **optimal** if there is a utility function  $u^h : \mathbb{R}_+^{S+1} \rightarrow \mathbb{R}$  solving

$$u^h(x^h(q, e^h)) = \sup u^h(e^h + W\mathbb{R}^J).$$

**Definition 1.**  $(q, e) \in Q \times \Omega$  is an **equilibrium** if trades clear,  $\Sigma \theta^h(q, e^h) = 0$ . It is a **no-trade equilibrium** if  $\theta^h(q, e^h) = 0$  for every  $h$ .

We denote by  $\mathbb{E}, \mathbb{T}, \mathbb{X}$  the sets of equilibria, no-trade equilibria, and equilibrium allocations: an equilibrium allocation is any  $x$  for which  $(q, x) \in \mathbb{T}$  for some  $q \in Q$ .

### 2.1. Assumptions

**Assumption 1.** In the economy,  $e^h \gg 0$  is interior and  $a \in \mathbb{R}^{S \times J}$  has rank  $J$ .

**Assumption 2.** Trades by  $h$  are optimal with respect to some utility  $u^h$ .

**Assumption 3.**  $u^h$  is continuous,  $\mathcal{C}^2$  in  $\mathbb{R}_{++}^{S+1}$ , strictly increasing ( $\forall x \in \mathbb{R}_{++}^{S+1}, Du^h(x) \gg 0$ ), strictly concave ( $\forall x \in \mathbb{R}_{++}^{S+1}, D^2u^h(x)$  is negative definite), and boundary averse ( $\forall x' \in \mathbb{R}_{++}^{S+1}, u^h(x) \geq u^h(x') \Rightarrow x \in \mathbb{R}_{++}^{S+1}$  and  $\lim_{x_s \rightarrow 0^+} D_{x_s}u^h(x) = \infty, \forall s$ ).

An instrumental notion is each household's  $\nabla^h$  (**marginal rates of substitution**), the row-vector function  $\mathbb{R}_{++}^{S+1} \rightarrow \mathbb{R}_{++}^{S+1}$ :

$$\nabla^h(x) := \left( \dots, \frac{D_{x_s}u^h(x)}{D_{x_0}u^h(x)}, \dots \right). \tag{4}$$

The reason is the well-known implication that the optimal asset trade  $\theta^h(q, e^h)$  is  $\mathcal{C}^1$ , and is characterized as the unique solution of

$$\nabla^h a - q = 0 \tag{\theta}$$

while evaluating  $\nabla^h$  at  $e^h + W\theta^h$ .

<sup>3</sup> All vectors are column vectors, unless stated otherwise.

### 3. Equilibrium allocations characterized

We characterize equilibria as solutions of the program  $\max_{\Sigma x^h=r} W_{\delta,\mu}(x)$  for some parametric social welfare functions  $W_{\delta,\mu}$ , where the parameters satisfy a specific restriction  $(\delta, \mu) \in \mathbb{P} \subset \mathbb{R}^H \times \mathbb{R}^{HS}$ . The social welfare function considered is

$$W_{\delta,\mu}(x) := \Sigma \delta^h u^h(x^h) - \Sigma \bar{\mu}^h \cdot x^h \tag{5}$$

where the parameters  $\bar{\mu}^h := (0, \mu^h) \in \mathbb{R}^{S+1}$  simply depend on  $\mu$ .

Key to the characterization is a function  $(\delta(x), \mu(x))$  from the set of resource distributions  $\Omega$  to the ambient space  $\mathbb{R}^H \times \mathbb{R}^{HS}$ , with typical  $h$  element

$$\delta^h(x) := \frac{t}{D_0 u^h} \quad \mu^h(x) := t(\nabla^h - \bar{\nabla}) \tag{6}$$

where

$$t = \Sigma D_0 u^h \in \mathbb{R}_{++} \quad \bar{\nabla} = \frac{1}{H} \Sigma \nabla^h \in \mathbb{R}_{++}^S \tag{7}$$

by **Assumption 3**, and the dependence of  $D_0 u^h$ ,  $\nabla^h$ ,  $\bar{\nabla}$ , and  $t$  on  $x$  is clear, and thus omitted.

In a nutshell, the logic of the characterization has two steps. In the “necessity step” (**Proposition 1**), we show that, if  $x^* \gg 0$  is an equilibrium allocation, then it solves the social welfare program for  $(\delta, \mu) = (\delta(x), \mu(x))$ , evaluated at  $x^*$ . Then, it is natural to derive from the first-order conditions of the maximum a set of parametric restrictions for  $(\delta, \mu)$ , which we can use to define the parameter set  $\mathbb{P} := \mathbb{D} \times \mathbb{M} \subset \mathbb{R}^H \times \mathbb{R}^{HS}$  independently of the social welfare function (5) or the functions (6), as follows:

$$\mathbb{D} := \left\{ \delta \in \mathbb{R}_{++}^H \mid \Sigma \frac{1}{\delta^h} = 1 \right\} \quad \mathbb{M} := \left\{ \mu \in (a^\perp)^H \mid \Sigma \mu^h = 0 \right\}. \tag{8}$$

In the “sufficiency step” (**Proposition 2**), we show that an allocation  $x^*$  solving the social welfare program for some  $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$  is an equilibrium allocation, and in turn  $(\delta, \mu)$  must be  $(\delta(x^*), \mu(x^*))$ , the value of (6) at the maximum. We include the proofs in the body of the paper because they are insightful and simple.

**Proposition 1 (Necessity).** *If  $x \gg 0$  is an equilibrium allocation, then*

- (A) *it solves the program  $\max_{\Sigma x^h=r} W_{\delta,\mu}(x)$  with  $(\delta, \mu) = (\delta(x), \mu(x))$  in (6),*
- (B)  *$(\delta(x), \mu(x))$  satisfy the restrictions  $\delta(x) \in \mathbb{D}$ ,  $\mu(x) \in \mathbb{M}$ .*

**Proof.** (A) Feasibility:  $\Sigma x^h = r$  holds because  $x$  is an equilibrium allocation. Maximality: by Kuhn–Tucker theorem, it suffices that  $x$  maximizes

$$W_{\delta,\mu}(x) - \rho \cdot \Sigma x^h \tag{9}$$

for some  $\rho \in \mathbb{R}^{S+1}$ . Let us show that this is true for  $\rho := t(1, \bar{\nabla})$ , with  $t, \bar{\nabla}$  as in (7). Since (9) is a concave function, given  $\delta = \delta(x) \gg 0$ , the equilibrium allocation  $x \gg 0$  is a maximizer as long as it makes its derivative equal zero. Its derivatives, at the assumed  $\rho$ , are

$$\begin{aligned} x_0^h &: \delta^h D_0 u^h - t \\ x_1^h &: \delta^h D_1 u^h - \mu^h - t \bar{\nabla} \end{aligned}$$

where  $\mathbf{1} = \{1, \dots, S\}$ ,  $x_1 := (x_1, \dots, x_S)$ . By (6) and the assumption that  $\delta = \delta(x) \gg 0$ , the first derivative is zero. By the assumption that  $(\delta, \mu) = (\delta(x), \mu(x))$  in (6), we can write the derivatives with respect to  $x_1^h$  as

$$\delta^h D_1 u^h - \mu^h - t \bar{\nabla} = \frac{t}{D_0 u^h} D_1 u^h - t(\nabla^h - \bar{\nabla}) - t \bar{\nabla}$$

which, by definition of  $\nabla^h$  in (4), equals zero.

(B) We show that  $\delta = \delta(x) \in \mathbb{D}$ ; observe that  $\Sigma(1/\delta^h) = \Sigma(D_0 u^h/t) = 1$ , by (6) and (7), and that  $\delta > 0$ , by **Assumption 2**. Next, we show that  $\mu \in \mathbb{M}$ .  $\mu = \mu(x) = t(\nabla^h - \bar{\nabla})$ ; summing over consumers, and using definition (7),  $\Sigma \mu^h = t(\Sigma \nabla^h - H \bar{\nabla}) = 0$ .  $x \gg 0$ , and **Assumption 2**, imply that  $t = \Sigma D_0 u^h > 0$ . Finally, using the fact that  $x \gg 0$  is an equilibrium allocation (precisely, individually optimal in the sense of  $(\theta)$ , implies that  $\mu \in (a^\perp)$ . ■

**Proposition 2 (Sufficiency).** *If  $x > 0$  solves the program  $\max_{\Sigma x^h=r} W_{\delta,\mu}(x)$  for some  $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$ , then*

- (A) *it is an equilibrium allocation,*
- (B)  *$(\delta, \mu) = (\delta(x), \mu(x))$  in (6); in particular,  $x \gg 0$ .*

**Proof.** (B):  $x = \arg\{\max_{\Sigma x^h=r} W_{\delta,\mu}(x)\} \cap \mathbb{R}_{++}^{H(S+1)}$ , for some  $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$ , follows from the boundary aversion in Assumption 2, and  $\delta \gg 0$ . Next, since  $W_{\delta,\mu}(x)$  is concave and the constraint  $\Sigma x^h = r$  is linear, Kuhn–Tucker's constraint qualification holds. By the Kuhn–Tucker theorem,  $x$  maximizes  $W_{\delta,\mu}(x) - \rho \cdot \Sigma x^h$ , for some  $\rho \in \mathbb{R}^{S+1}$ . Since  $x \gg 0$ , the maximum is characterized by its first-order conditions,

$$\begin{aligned} x_0^h &: \delta^h D_0 u^h - \rho_0 = 0 \\ x_1^h &: \delta^h D_1 u^h - \mu^h - \rho_1 = 0 \end{aligned}$$

$\forall h$ . Hence,  $\delta^h = \rho_0 / D_0 u^h$ . This and the hypothesis that  $\Sigma \frac{1}{\delta^h} = 1$ , implicit in  $\delta \in \mathbb{D}$ , imply that  $\rho_0 = \Sigma D_0 u^h > 0$ ,

$$\delta^h = \frac{\rho_0}{D_0 u^h} = \delta(x).$$

Rewriting the second set of first-order conditions,

$$\mu^h = \delta^h D_1 u^h - \rho_1 = (\Sigma D_0 u^i) \nabla^h - \rho_1.$$

This and the hypothesis  $\mu \in \mathbb{M}$  (so that  $\Sigma \mu^h = 0$ ) imply that  $0 = \Sigma \mu^h = (\Sigma D_0 u^i) \Sigma \nabla^h - H \rho_1$ ; hence  $\rho_1 = (\Sigma D_0 u^i) \bar{\nabla}$ . Substituting back,

$$\mu^h = (\Sigma D_0 u^i) (\nabla^h - \bar{\nabla}),$$

yielding  $\mu = \mu(x)$  in (6).

(A): we are left to show that there exists an asset price  $q \in Q$  such that  $(q, x) \in \mathbb{T}$ , where from now on we refer to  $x \gg 0$  as a solution to the social welfare program. This is equivalent to finding a price  $q$  such that  $\theta^h(q, x) = 0$  for all  $h$ . Let  $q := \bar{\nabla} a$ , where  $\bar{\nabla}$  is evaluated at  $x$ . Then, by individual optimality ( $\theta$ ), it suffices to show that  $\nabla^h a - \bar{\nabla} a = 0$ , also evaluated at  $x$ . Indeed, we proved that, at  $x$ ,  $(\Sigma D_0 u^i) (\nabla^h - \bar{\nabla}) = \mu^h$ , which by hypothesis of  $\mu \in \mathbb{M}$  implies that  $(\nabla^h - \bar{\nabla}) \in a^\perp$  (the vector space  $a^\perp$  being closed under rescaling,  $\Sigma D_0 u^i \neq 0$ ). ■

Putting together part (A) in both Propositions 1 and 2 yields our characterization of equilibria:

**Theorem 1.**  $x \gg 0$  is an equilibrium allocation if and only if it solves  $\max_{\Sigma x^h=r} W_{\delta,\mu}(x)$  for some  $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$ .

We remark that if the asset markets are complete, then  $a^\perp = \{0\}$  and  $\mathbb{M} = 0$  and  $\mu = 0$  necessarily, making  $W_{\delta,\mu} = \Sigma \delta^h u^h(x^h)$ . In particular, if the asset markets are complete, Theorem 1 simply concludes that  $x$  is an equilibrium allocation if and only if it solves  $\max_{\Sigma x^h-r=0} W_{\delta,\mu} = \Sigma \delta^h u^h(x^h)$  for some  $\delta \in \mathbb{D}$ —the classical characterization.

As an aside, there is a separate characterization, which does not even refer to social welfare functions. The proof is relegated to the appendix, and simply recycles the arguments above.

**Corollary 1.** Suppose  $x \in \Omega$ . Then it is an equilibrium allocation if and only if  $(\nabla^h - \bar{\nabla})x \in a^\perp$  for every  $h$ .

#### 4. Dimension of the equilibrium allocations

Here, we state a version of the preceding results that is stronger, since it claims the above relation  $x \leftrightarrow (\delta, \mu)$  between the space of equilibrium allocations  $\mathbb{X}$  and the parameter space  $\mathbb{D} \times \mathbb{M}$  is a **diffeomorphism**—a bijection, smooth in both directions. If the utilities are time separable,  $u(x) = u_0(x_0) + u_1(x_1)$ , the inverse of the functions in (6) can be easily seen to determine the consumption allocation,  $(x_0, x_1)$ . Indeed, given (7), first,  $x_0$  is determined by inverting  $\delta(x_0)$ , then  $x_1$  is determined by inverting  $\mu(x)$  at  $x_0$ .

By way of caveat, the dimensions reported here are to be interpreted for fixed asset claims  $a$  and resources  $r$ .

**Theorem 2.** The no-trade equilibria  $\mathbb{T}$  are diffeomorphic to  $\mathbb{D} \times \mathbb{M}$ .<sup>4</sup> A global chart is (6). Thus  $\mathbb{T}$  is a smooth manifold of dimension  $(H-1)(S-J+1)$ .

To see why this is the dimension, note that  $\mathbb{T}$  and  $\mathbb{D} \times \mathbb{M}$  must have the same dimension, being diffeomorphic, and that the dimension of  $\mathbb{D} \times \mathbb{M} = \dim(\mathbb{D}) + \dim(\mathbb{M})$  is easy to compute. Glancing at (6), we see that  $\mathbb{D}$  is equivalent to a  $(H-1)$ -simplex; hence  $\dim(\mathbb{D}) = H-1$ . This would be the dimensionality of the equilibrium set if markets were complete, since then  $J = S$  implies that  $a^\perp = \{0\}$  and  $\mathbb{M} = 0$ . Otherwise, for  $J < S$ ,  $\dim(\mathbb{M}) = (H-1)(S-J)$ , because choosing  $\mu^1, \dots, \mu^{H-1}$  from  $a^\perp$  (itself of dimension  $S-J$ ) uniquely determines  $\mu^H \in a^\perp$  as  $\mu^H = -\Sigma_{h < H} \mu^h$ . Thus

$$\dim(\mathbb{T}) = \dim(\mathbb{D}) + \dim(\mathbb{M}) = (H-1) + (H-1)(S-J) = (H-1)(S-J+1).$$

Understanding the dimensionality helps also to appreciate how Theorem 2 nests a well-known fact about complete markets: the interior Pareto optima (which are  $\mathbb{X}$  by the two welfare theorems) have dimension  $H-1 = \dim(\mathbb{D})$ ; see proof 5.2.4 in Balasko (1988).

<sup>4</sup> Note that here the statement is about  $\mathbb{T}$  instead of  $\mathbb{X}$ , but this is equivalent, as shown by the inverses  $(q, x) \rightarrow x, x \rightarrow (\bar{\nabla}, x)$ .

**Corollary 2.** *The equilibria  $\mathbb{E}$  are a  $(H - 1)J$ -vector bundle on  $\mathbb{T}$ , and hence a smooth  $(H - 1)(S + 1)$ -manifold.*

To see why this is the dimension, note that  $\mathbb{E}$ , as the Cartesian product of  $\mathbb{T}$  and a vector space of dimension  $(H - 1)J$ , has dimension

$$\dim(\mathbb{E}) = \dim(\mathbb{T}) + (H - 1)J \stackrel{\text{Theorem 2}}{=} (H - 1)(S + 1).$$

Corollary 2 agrees with a well-known fact about complete markets: the equilibrium manifold, given fixed resources  $r$ , has dimension  $(H - 1)$  times the number of goods (i.e.  $S + 1$ , in a contingent market economy with a single spot per state  $s$ ); see Chapter 5 in Balasko (1988).

### 5. Proofs

We start from the well-known fact:<sup>5</sup> that

**Proposition 3.**  *$\mathbb{E}$  is a smooth manifold.*

Our argument applies the very useful lemma to the sets and maps which are defined in the sequel:

**Lemma 1** (3.2.1 in Balasko (1988)). *Let  $\phi : X \rightarrow Y$ ,  $\psi : Y \rightarrow X$  be smooth maps between smooth manifolds making  $\phi \circ \psi$  the identity. Then  $\psi(Y)$  is a smooth submanifold of  $X$  diffeomorphic to  $Y$ .*

$X$  is  $\mathbb{E}$   
 $Y$  is  $\mathbb{D} \times \mathbb{M}$ .

The maps are the following.  $\phi : \mathbb{E} \rightarrow \mathbb{D} \times \mathbb{M}$  is

$$\phi(q, e) = \left[ \begin{array}{c} \Sigma D_0 u^i \\ \dots, \frac{D_0 u^h}{D_0 u^h}, \dots \\ \dots, (\Sigma D_0 u^i)(\nabla^h - \bar{\nabla}), \dots \end{array} \right] \tag{10}$$

evaluated at consumptions  $e + W\theta(q, e)$ .  $\psi : \mathbb{D} \times \mathbb{M} \rightarrow \mathbb{E}$  is

$$\psi(\delta, \mu) = (\bar{\nabla}a, x) \tag{11}$$

where  $\bar{\nabla} := \frac{1}{H} \Sigma \nabla^h$  is the average of the  $\nabla^h(x)$  in (4) and

$$x := \arg \max_{x \in \bar{\Omega}} \Sigma(\delta^h u^h(x^h) - \bar{\mu}^h x^h) \tag{12}$$

where  $\bar{\mu}^h := (0, \mu^h)$ , and  $\bar{\Omega}$  is the closure of  $\Omega$ .

**Lemma 2.**  *$\phi$  and  $\psi$  are well defined and satisfy the hypothesis in Lemma 1.*

**Proof of Lemma 2.**  $\phi$  is well defined, i.e.  $\phi(q, e) \in \mathbb{D} \times \mathbb{M}$ . Clearly  $\phi_1 = (\dots, \frac{\Sigma D_0 u^i}{D_0 u^h}, \dots) \in \mathbb{D}$ . As for  $\phi_2$ , first  $\phi_2^h \in a^\perp$ : since equilibrium trades are optimal,  $\nabla^h a = q$ , which averaged implies that  $\bar{\nabla}a = q$ , which subtracted from  $\nabla^h a = q$  implies that  $(\nabla^h - \bar{\nabla})a = q - q = 0$  or  $\phi_2 = (\Sigma D_0 u^i)(\nabla^h - \bar{\nabla}) \in a^\perp$ . Second,  $\Sigma \phi_2^h = 0 : \Sigma \phi_2^h = (\Sigma D_0 u^i) \Sigma(\nabla^h - \bar{\nabla}) = (\Sigma D_0 u^i) \times 0 = 0$  because  $\bar{\nabla}$  is the average.

$\psi$  is well defined, i.e.  $\psi(\delta, \mu)$  exists, is unique, and in  $\mathbb{E}$ .  $x$  exists because the objective in (12) is continuous and  $\bar{\Omega}$  compact. By boundary aversion in Assumption 3,  $x \gg 0$ ; moreover, it is unique because Assumption 3 implies that the objective is strictly concave in the interior. We now show that  $(\bar{\nabla}a, x) \in \mathbb{E}$ , by showing  $(\bar{\nabla}a, x) \in \mathbb{T}$ , i.e.  $\theta^h(\bar{\nabla}a, x^h) = 0$ . By  $(\theta)$ , we must show that

$$\nabla^h a - \bar{\nabla}a = 0 \tag{*}$$

while evaluating  $\nabla^h$  at  $x^h + W0 = x^h$ . Since  $x$  is defined by (12), Kuhn–Tucker theorem implies that there exists  $\rho_+ = (\rho_0, \rho) \in \mathbb{R}^{S+1}$  such that it maximizes

$$\Sigma(\delta^h u^h(x^h) - \bar{\mu}^h x^h) - \rho'_+ \Sigma x^h.$$

So

$$\delta^h D u^h - \bar{\mu}^h = \rho'_+. \tag{**}$$

<sup>5</sup> Geanakoplos and Polemarchakis (1986), Section 6.

Eq. (\*\*) implies for state 0 that

$$\delta^h = \frac{\rho_0}{D_0 u^h}$$

since  $\bar{\mu}_0^h = 0$ , and this in turn with  $\delta \in \mathbb{D}$  (so that  $\Sigma \frac{1}{\delta^h} = 1$ ) implies that  $\rho_0 = \Sigma D_0 u^i$ ; hence

$$\delta^h = \frac{\Sigma D_0 u^i}{D_0 u^h}. \quad (13)$$

And Eq. (\*\*) implies for states  $\mathbf{1} = \{1, \dots, S\}$  that

$$\mu^h = \delta^h D u^h - \rho' = (\Sigma D_0 u^i) \nabla^h - \rho'$$

on applying (13) and definition (4). Taking the average and using  $\mu \in \mathbb{M}$  reveals that  $0 = \frac{1}{H} \Sigma \mu^h = (\Sigma D_0 u^i) \bar{\nabla} - \rho'$  or  $\rho' = (\Sigma D_0 u^i) \bar{\nabla}$ , which substituted back implies that

$$\mu^h = (\Sigma D_0 u^i) (\nabla^h - \bar{\nabla}). \quad (14)$$

Since  $\mu \in \mathbb{M}$ , we have  $\mu^h \in a^\perp$ . So (14) divided by  $\Sigma D_0 u^i \neq 0$  implies that  $(\nabla^h - \bar{\nabla}) \in a^\perp$ , i.e.  $(\nabla^h - \bar{\nabla})a = 0$ , which is the desired (\*).

$\phi \circ \psi = id$ . Given  $(\delta, \mu)$ , define  $(q, x) := \psi(\delta, \mu)$ ; we want  $\phi(q, x) = (\delta, \mu)$ . We have seen that  $(\delta, \mu)$  is the right side of (13) and (14), evaluated at consumption  $x$ . Also, the right side of (13) and (14), evaluated at consumption  $x + W\theta(q, x)$ , is the definition of  $\phi(q, x)$ , for any  $(q, x)$ . So the right sides agree, and therefore  $(\delta, \mu) = \phi(q, x)$ , if both consumption agree, i.e. if  $\phi(q, x) = 0$ , which holds since  $(q, x) = \psi(\delta, \mu) \in \mathbb{T}$ , as just argued above.

*Smoothness.* That  $\phi$  is  $\mathcal{C}^1$  follows from its definition and that  $u^h$  is  $\mathcal{C}^2$ ,  $\theta^h$  is  $\mathcal{C}^1$ . That  $\psi$  is  $\mathcal{C}^1$  follows from the implicit function theorem by a standard argument that we omit. ■

Now we prove Theorem 2.

**Proof of Theorem 2.** Lemma 2 verifies the hypothesis of Lemma 1, so we deduce that  $\psi(\mathbb{D} \times \mathbb{M})$  is a smooth manifold diffeomorphic to  $\mathbb{D} \times \mathbb{M}$ , and  $\phi|_{\psi(\mathbb{D} \times \mathbb{M})} : \psi(\mathbb{D} \times \mathbb{M}) \rightarrow \mathbb{D} \times \mathbb{M}$  is a diffeomorphism. Suppose for a moment that  $\psi(\mathbb{D} \times \mathbb{M}) = \mathbb{T}$ . Then we have deduced that  $\mathbb{T}$  is a smooth manifold diffeomorphic to  $\mathbb{D} \times \mathbb{M}$ , and  $\phi|_{\mathbb{T}} : \mathbb{T} \rightarrow \mathbb{D} \times \mathbb{M}$  is a diffeomorphism, i.e. a global chart, completing the proof of Theorem 2. (For the proof of the statement about  $\mathbb{T}$ 's dimension, see the paragraph just after the statement of Theorem 2.)

We now justify our momentary supposition that  $\psi(\mathbb{D} \times \mathbb{M}) = \mathbb{T}$ .

The proof of Lemma 2 shows that  $\psi(\mathbb{D} \times \mathbb{M}) \subset \mathbb{T}$  (where  $\psi$  is shown well defined), so we show that  $\mathbb{T} \subset \psi(\mathbb{D} \times \mathbb{M})$ , by showing that  $id_{\mathbb{T}} = \psi \circ \phi|_{\mathbb{T}}$ . Fix  $(q, e) \in \mathbb{T}$ . Write  $(\delta, \mu) := \phi(q, e)$ . We want  $\psi(\delta, \mu) = (q, e)$ . That is, what we want (by definition of  $\psi$ ) is (i)  $q = \bar{\nabla}|_e a$  and (ii)  $e = \arg \max_{x \in \bar{\Omega}} \Sigma (\delta^h u^h(x^h) - \bar{\mu}^h x^h)$ .

We show (i). We know that Eq. ( $\theta$ ) holds at any equilibrium allocation. Since  $(q, e) \in \mathbb{T}$ ,  $e + W0 = e$  is a no-trade equilibrium allocation. Therefore Eq. ( $\theta$ ) holds at  $e : q = \nabla^h|_{e^h} a$ . Averaged, this implies that  $q = \bar{\nabla}|_e a$ .

We show (ii). Since  $(q, e) \in \mathbb{T}$ ,  $e$  is an equilibrium allocation, so by part (A) of Proposition 1,  $e = \arg \max_{x \in \bar{\Omega}} W_{\delta, \mu}(x)$  for  $(\delta, \mu) := \phi(q, e)$ . ■

Finally, we provide the proof of Corollary 2.

**Proof of Corollary 2.** The projection  $\mathbb{E} \rightarrow \mathbb{T}$ ,  $\pi(q, e) = (q, x(q, e))$  is well defined; clearly, its fibers are

$$\pi^{-1}(q, x) = \{q\} \times \{e \in \Omega : \forall h, e^h = x^h - y^h \text{ for some } y^h \in \text{span}(W)\}.$$

So fibers are parametrized by an open set of  $(y^2, \dots, y^H)$  in  $\text{span}(W)^{H-1}$  – here  $e^1 = x^1 + \Sigma_{h>1} y^h$  – which is a convex set of dimension  $(H-1)J$ , depending smoothly on  $(q, e)$ . ■

**Proof of Corollary 1.** Suppose  $x \in \Omega$ ; hence  $x \gg 0$ . If it is an equilibrium allocation, then all  $(\nabla^h - \bar{\nabla})|_x \in a^\perp$  according to part (B) of Proposition 1. Conversely, if  $(\nabla^h - \bar{\nabla})|_x \in a^\perp$  for all  $h$ , we have  $x$  to be an equilibrium allocation, i.e. we need an asset price  $q \in Q$ , such that  $(q, x) \in \mathbb{T} \iff \theta^h(q, x) = 0$  for all, but one,  $h$ . The choice  $q := \bar{\nabla}|_x$  works, by the same argument used to prove part (A) of Proposition 2. ■

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